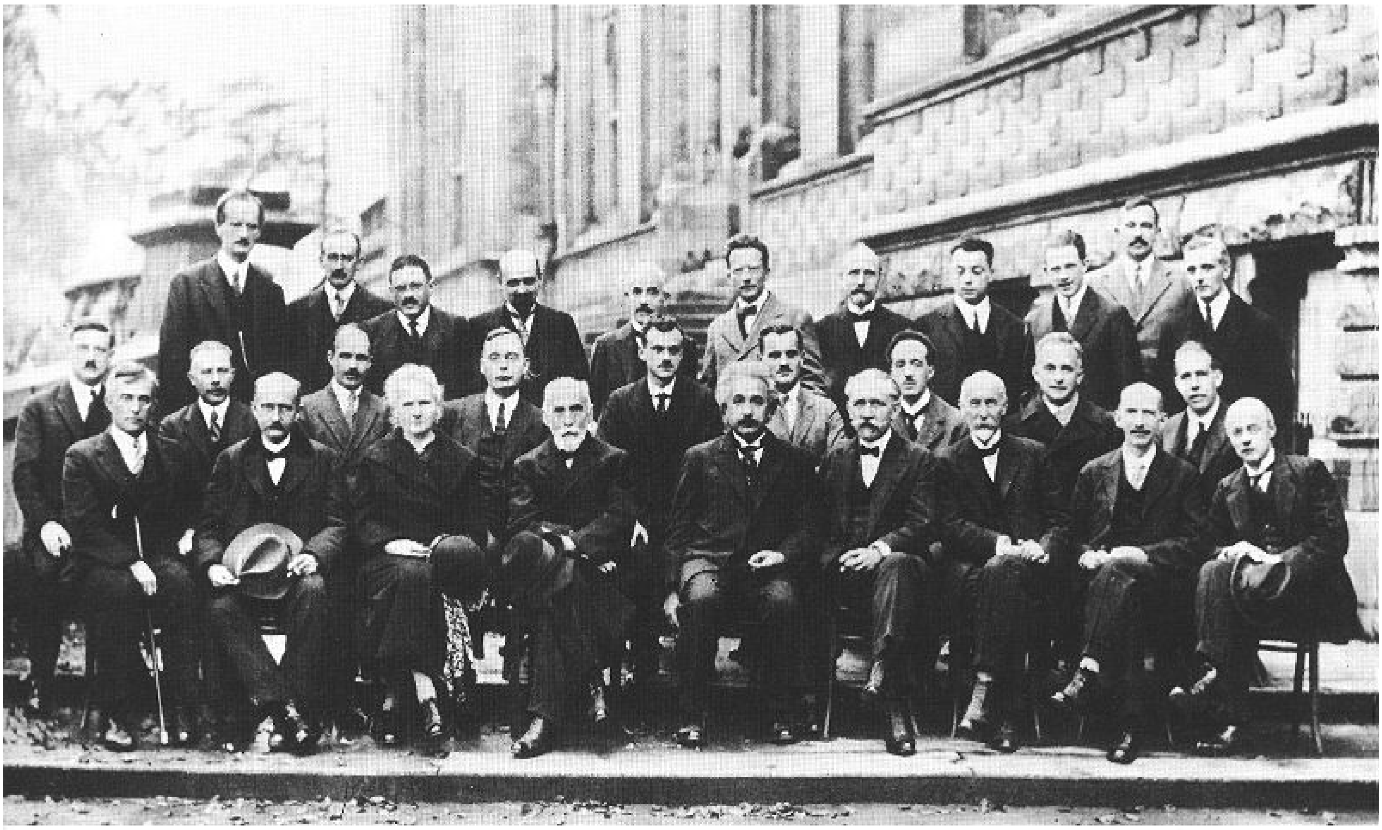
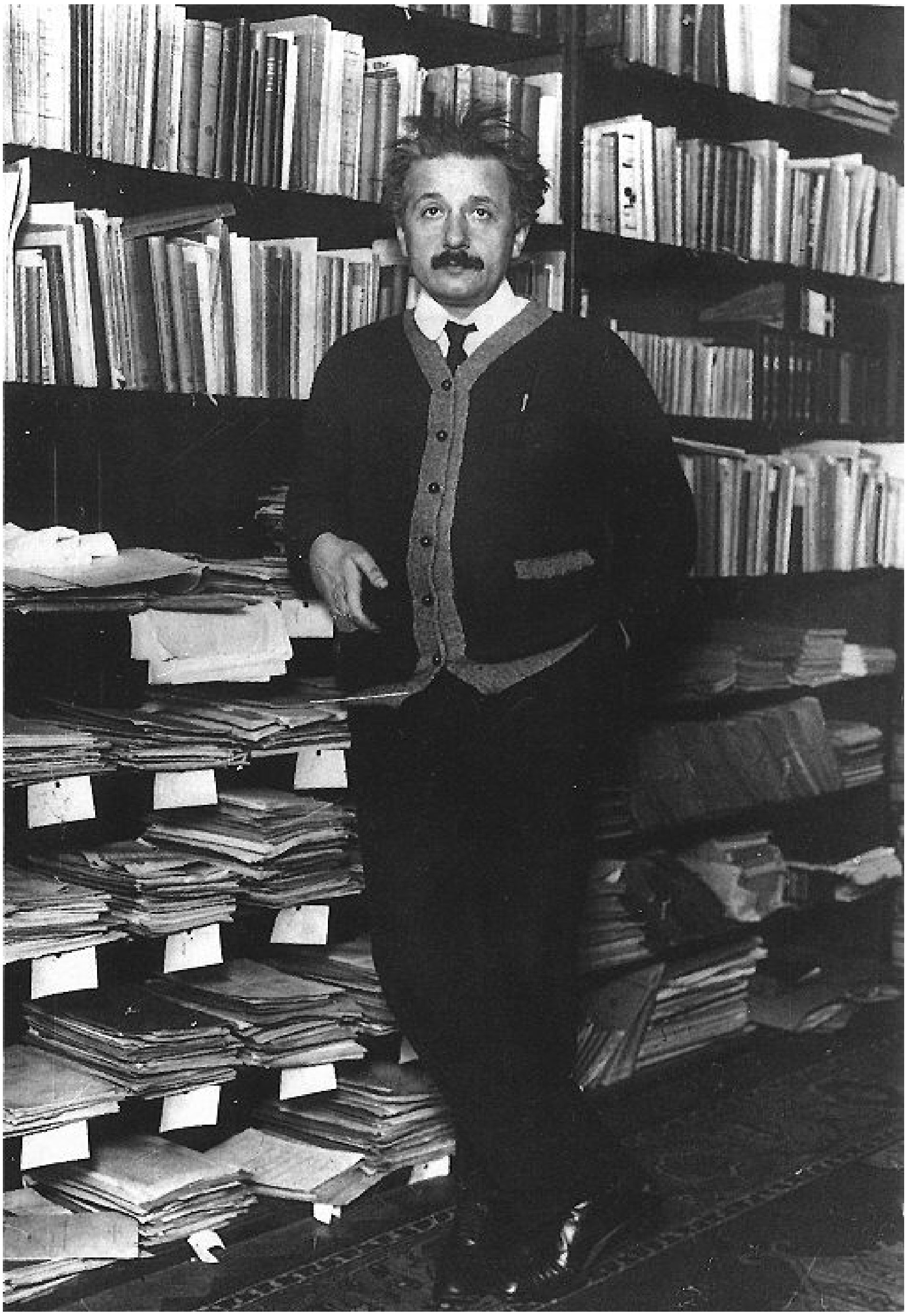
TESTING CAUSAL QUANTUM MECHANICS BY JOINT MEASUREMENTS ON ENTANGLED COHERENT STATES WITH LARGE UNCERTAINTY PRODUCT

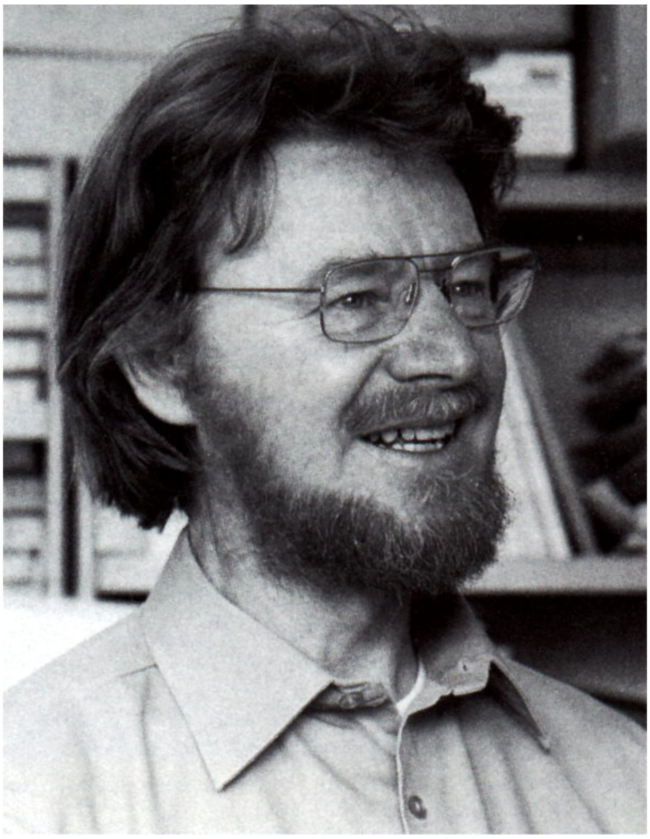
S. M. ROY

HBCSE, TIFR



A. PICCARD E. HENRIOT P. EHRENFEST Ed. HERZEN Th. DE DONDER E. SCHRODINGER E. VERSCHAFFELT W. PAULI W. HEISENBERG R.H. FOWLER L. BRILLOUIN
P. DEBYE M. KNUDSEN W.L. BRAGG H.A. KRAMERS P.A.M. DIRAC A.H. COMPTON L. de BROGLIE M. BORN N. BOHR
I. LANGMUIR M. PLANCK Mme CURIE H.A. LORENTZ A. EINSTEIN P. LANGEVIN Ch.E. GUYE C.T.R. WILSON O.W. RICHARDSON

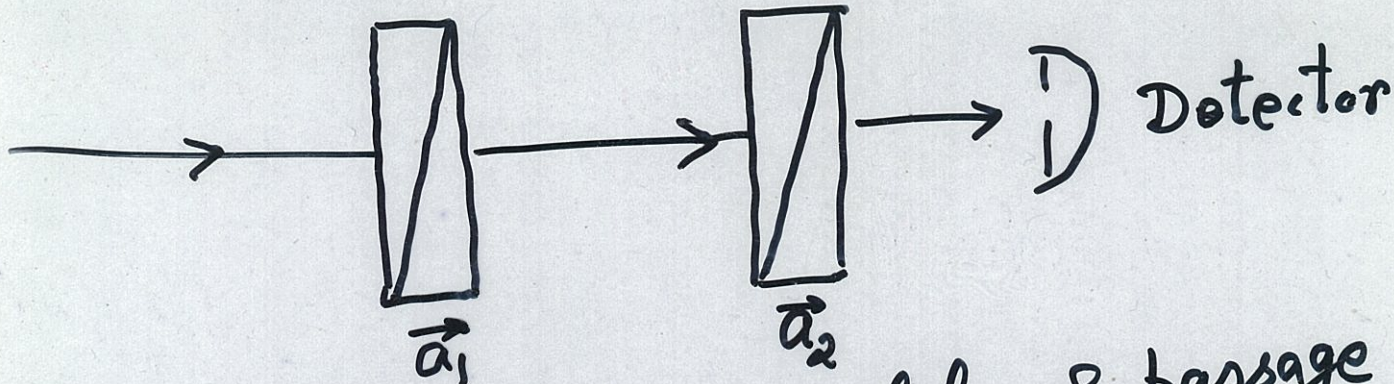




DISCOMFORT WITH QUANTUM MECHANICS

- NO CAUSALITY (INDETERMINISM)
- INTRINSICALLY APPROXIMATE
- QUANTUM STATE DOES NOT HAVE IMAGE OF "REAL WORLD"
- NO TRAJECTORIES
- CONTEXTUALITY: GLEASON-KOCHEN-SPECKER & BELL THEOREMS

WHAT IS CAUSALITY?



$\vec{a}_1 \cdot \vec{a}_2 = \cos\theta$, $\cos^2\theta$ = probab. of passage thro' detector

DIFFERENT RESULTS HAVE DIFFERENT CAUSES : Photons which pass thro' have different properties from those which dont : NOT REFLECTED IN $|\psi\rangle$

"I AM, IN FACT, RATHER FIRMLY CONVINCED THAT THE ESSENTIALLY STATISTICAL CHARACTER OF CONTEMPORARY QUANTUM THEORY IS SOLELY TO BE ASCRIBED TO THE FACT THAT THIS (THEORY) OPERATES WITH AN INCOMPLETE DESCRIPTION OF PHYSICAL SYSTEMS"

— A. EINSTEIN

CLASSICAL MECHANICS IS CAUSAL

(THIS INCLUDES CLASSICAL WAVE MECHANICS :
difference, e.g. classical E.M. WAVE FIELDS
have a direct physical meaning)

CAUSALITY AND LOCAL REALITY IN THE CLASSICAL WORLD

State: $\{x_i, p_i \mid i = 1, \ldots 3n\}_{t=0}$

$$\xrightarrow[\text{Relativistic Version}]{\text{Newton's Eqns}} \{x_i', p_i' \mid i = 1, \ldots 3n\}_t$$

PHASE-SPACE TRAJECTORIES

CAUSALITY: Different results arise from different causes
(JAUCH)

OBJECTIVE REALITY: $\vec{x}, \vec{p}$ are properties indep. of OBSERVATION

LOCALITY: x_i, p_i independent of (not influenced by) actions at space-like separation.

J.S.BELL (TIFR 1980 OCT)

What in the world is quantum mechanics about exactly ?

QM is about:

Wavefunctions ψ

Operators **H**

Schroedinger eq:

$$i(h/2\pi)\partial\psi/\partial t = H\psi$$

and how to solve it.

But what in the world ?

The waveunction is not like the world:

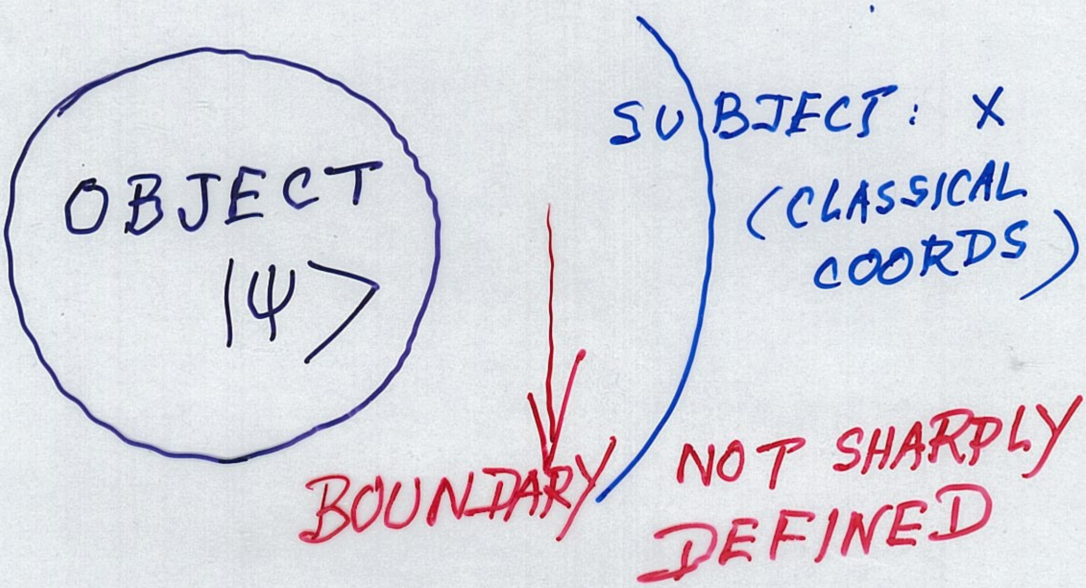
$$\psi=\phi+\chi$$

would be unrecognizable.

So: Statistical Interpretation
Statistics of what? 'measurement results'

'MEASUREMENT' PRESUPPOSES

SUBJECT - OBJECT SPLIT OF THE WORLD (BOHR, HEISENBERG)



QM ASSIGNS PROBABS. OF 'A' BEING OBSERVED TO BE 'a':

$|\langle a|\psi\rangle|^2$, where $A|a\rangle = a|a\rangle$

<u>NOT</u> PROBABS. OF 'A' being 'a',

e.g. NO JOINT PROBABS. FOR q, p (eigenvalues of non-commuting observables)

BELL: QM INTRINSICALLY APPROXIMATE, INTRINSICALLY AMBIGUOUS BECAUSE SUBJECT-OBJECT BOUNDARY CANNOT BE SHARPLY DEFINED.

J.S.BELL ,CERN PREPRINT
"Towards an exact quantum mechanics"

Fundamental ambiguity:

Nobody knows what quantum mechanics says

Exactly about any situation.

For nobody knows where the boundary really is,

between wavy quantum system and the world of

particular events.

THIS IS THE PROBLEM
OF QUANTUM MECHANICS

It is no problem in practice-
because practice is not accurate enough-
and maybe never will be.

A Q.M. WITHOUT OBSERVERS, AND THEREFORE WITHOUT THIS FUNDAMENTAL AMBIGUITY WILL NEVERTHELESS BE A BIG ADVANCE IN PRINCIPLES OF Q.M.

APPROACHES TO QM OF CLOSED SYSTEMS

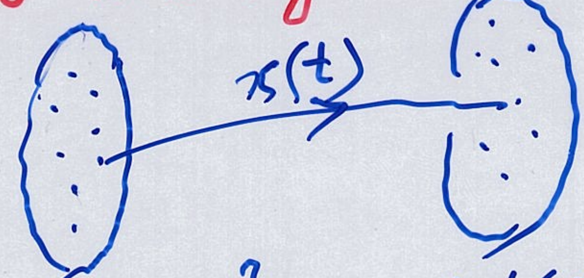
Many Universe: EVERETT (1957)
Wheeler (1957), De Witt (1970)

Consistent Histories: Gell-Mann & Hartle, Griffiths, Omnes

Causal Q.M: De Broglie (1926), Bohm (1952)

Bell: The De Broglie - Bohm picture disposes of the necessity to divide the world somehow into system and apparatus

State $(|\psi\cancel{(t)\rangle}, x(t))$
$(|\psi(t)\rangle, x(t))$



Probab. distbn: $dx\,|\langle x|\psi(t=0)\rangle|^2 \longrightarrow dx\,|\langle x|\psi(t)\rangle|^2$
due to law of motion for $x(t)$

Position- Momentum Symmetric Causal Q.M:
S.M. Roy and V. Singh (1995)
State $(|\psi(t)\rangle, x(t), p(t))$
corresponding Observables Non-commuting!
APPROX. JT. MEASUREMENT POSSIBLE.

ONE OF THE EARLIEST TRAJECTORY DESCRIPTIONS : DE BROGLIE - BOHM

Joint probab. density

$$\rho_{dBB}(x,p,t) = |\langle x|\psi(t)\rangle|^2 \, \delta(p - \nabla S),$$

$$\psi = |\psi| \, e^{iS/\hbar}$$

Satisfies $\int \rho(x,p,t)\,dp = |\psi(x,t)|^2$

Unsatisfactory because (Ref. TAKABAYASHI)

$$\int \rho_{dBB}(x,p,t)\,dx \neq |\langle p|\psi(t)\rangle|^2$$

E.g. for the minm. uncertainty state

$$\langle x|\psi\rangle = \frac{1}{\sqrt{\sigma}} \frac{1}{(2\pi)^{1/4}} e^{ixp_0 - \frac{1}{4\sigma^2}(x_0 - x)^2}$$

we get,

$$\int \rho_{dBB}(x,p,t)\,dx = \delta(p - p_0)$$

$$\therefore (\Delta p)_{dBB} = 0,$$

whereas

$$(\Delta p)_{QM} = \frac{1}{2\sigma}$$

WIGNER (1932)

$|\langle\vec{x}|\psi\rangle|^2$ and $|\langle\vec{p}|\psi\rangle|^2$ as marginals of a joint distribution

$W(\vec{x},\vec{p})$: for $\vec{x} = (x_1, \dots, x_n)$,

$$W(\vec{x},\vec{p}) = \int_{-\infty}^{\infty} \frac{d\vec{y}}{(2\pi)^n} \langle \vec{x} - \frac{\vec{y}}{2}|\psi\rangle\langle\psi|\vec{x} + \frac{\vec{y}}{2}\rangle \times e^{i\vec{p}\cdot\vec{y}},$$

$$\int W(\vec{x},\vec{p})\, d\vec{p} = |\langle\vec{x}|\psi\rangle|^2$$

$$\int W(\vec{x},\vec{p})\, d\vec{x} = |\langle\vec{p}|\psi\rangle|^2$$

BUT W NOT POSITIVE DEFINITE!

$$\int d\vec{x}\, d\vec{p}\; W_\psi(\vec{x},\vec{p})\, W_\phi(\vec{x},\vec{p}) = \frac{1}{(2\pi)^n} |\langle\psi|\phi\rangle|^2$$

IS THERE A (POSITIVE) JOINT PROBAB. DISTRIBUTION RETURNING Q.M. PROBABS. AS MARGINALS?

(SMEARED WIGNER FUNCTIONS SUCH AS THE HUSIMI FUNCTION, ALTHOUGH POSITIVE, DO NOT REPRODUCE $|\langle\vec{x}|\psi\rangle|^2$, $|\langle\vec{p}|\psi\rangle|^2$ EXACTLY)

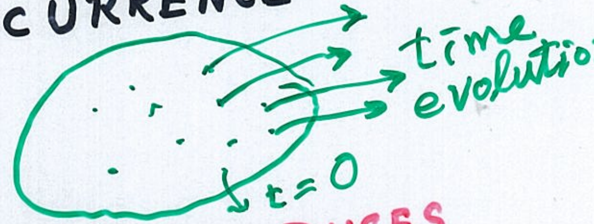
TOWARDS REALISTIC (CAUSAL) Q.M. SYMMETRIC IN x, p:

EXISTENCE OF DEFINITE $\vec{x}(t)$, $\vec{p}(t)$ FOR INDIVIDUAL SYSTEM NOT IN CONFLICT WITH Q.M. BECAUSE ONLY ENSEMBLES OF THEM HAVE TO CORRESPOND TO A STATE VECTOR $|\psi(t)\rangle$ AND OBEY THE UNCERTAINTY PRINCIPLE.

SUPPOSE INDIVIDUAL STATE $= \left\{ |\psi(t)\rangle, \vec{x}(t), \vec{p}(t) \right\}$

WITH PROBAB. OF OCCURRENCE

$$\rho_\psi(\vec{x}, \vec{p}, t)$$



CAN WE FIND ρ WHICH REPRODUCES Q.M. RESULTS ?

(e.g. $|\langle \alpha | \psi(t) \rangle|^2$, α: EIGENVALUES OF ANY CCS (COMPLETE COMMUTING SET) OF OBSERVABLES).

E. P. WIGNER (1932):

$$\rho_W(\vec{x}, \vec{p}, t) = \int_{-\infty}^{\infty} \frac{d\vec{y}}{(2\pi\hbar)^n} \langle \vec{x} - \tfrac{\vec{y}}{2} | \psi(t) \rangle \langle \psi(t) | \vec{x} + \tfrac{\vec{y}}{2} \rangle e^{\frac{i\vec{p}\vec{y}}{\hbar}}$$

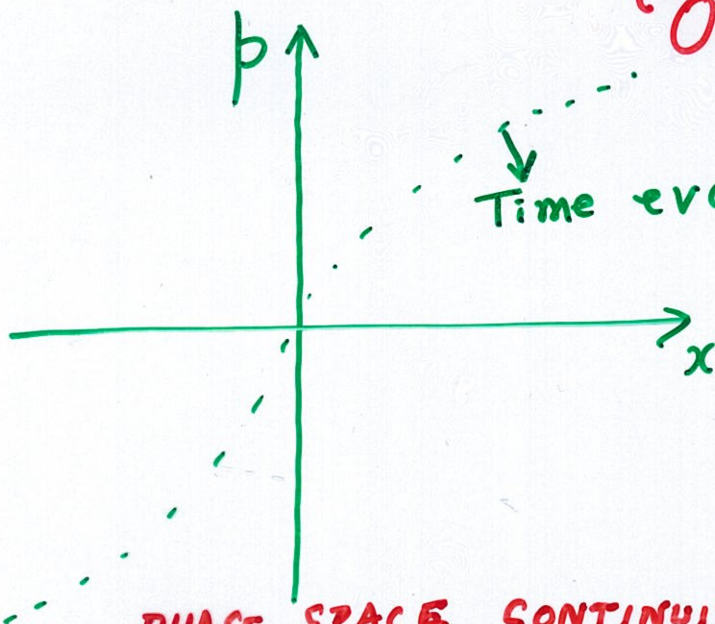
DOES THE JOB FOR $\alpha = x_1, x_2;\ p_1, p_2;\ x_1, p_2;\ p_1, x_2$

BUT ρ_W NOT POSITIVE DEFINITE!

GOOD NEWS: NON-UNIQUE SIMPLE SOLN.

PRESENT PROPOSAL: (SMR, VS) (1995)

'ORGANIZING PRINCIPLE'



Time evolution: $H_{C,\psi}(x,p,t)$

$$H_{cl} \rightarrow H \rightarrow \psi(x,t)$$

$$\longrightarrow H_{C,\psi}(x,p,t)$$

PHASE SPACE CONTINUITY EQN:

$$\frac{\partial \rho(x,p,t)}{\partial t} + \frac{\partial}{\partial x}(\rho \underset{\downarrow\ \frac{\partial H_c}{\partial p}}{\dot{x}}) + \frac{\partial}{\partial p}(\rho \underset{\downarrow\ -\frac{\partial H_c}{\partial x}}{\dot{p}}) = 0$$

HAMILTONIAN FLOW $\Rightarrow$ LIOUVILLE CONDITION

$$\frac{\partial \rho}{\partial t} + \dot{x}\frac{\partial \rho}{\partial x} + \dot{p}\frac{\partial \rho}{\partial p} = 0$$

SEEK $\rho(x,p,t)$ SUCH THAT:

1. $\rho \geqslant 0$
2. $\int \rho(x,p,t)\,dp = |\psi(x,t)|^2$
3. $\int \rho(x,p,t)\,dx = |\tilde{\psi}(p,t)|^2$
4. $d\rho/dt = 0$.

SIMPLEST ANSATZ: $\rho = |\psi(x,t)|^2\,\delta(p - \hat{p}(x,t))$

Motivation For δ-function Ansatz:

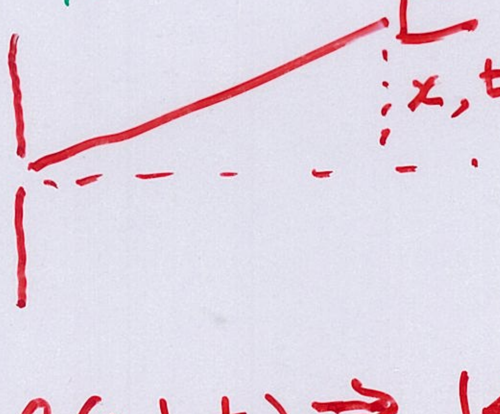
FREE Particle

$$\overbrace{\langle p|\psi(t)\rangle}^{\tilde{\psi}(p,t)} = \frac{1}{\sqrt{2\pi}}\int dx\; e^{-i\frac{p^2}{2m}t - ipx}\,\psi(x,t=0)$$

$$|\langle p|\psi(t)\rangle| = |\langle p|\psi(0)\rangle|$$

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}}\int dp\; e^{ipx}\, e^{-i\frac{p^2t}{2m}}\,\tilde{\psi}(p,0)$$
$$\frac{1}{\sqrt{2\pi}}\int dp'\; e^{-ip'x}\, e^{-i p'^2\frac{t}{2m}}\,\tilde{\psi}(p',0)^*$$

$$\frac{t}{m}\left|\psi\left(x=\frac{pt}{m},t\right)\right|^2 \xrightarrow[\text{(Stationary Phase Approx)}]{t\to\infty} |\tilde{\psi}(p,0)|^2$$



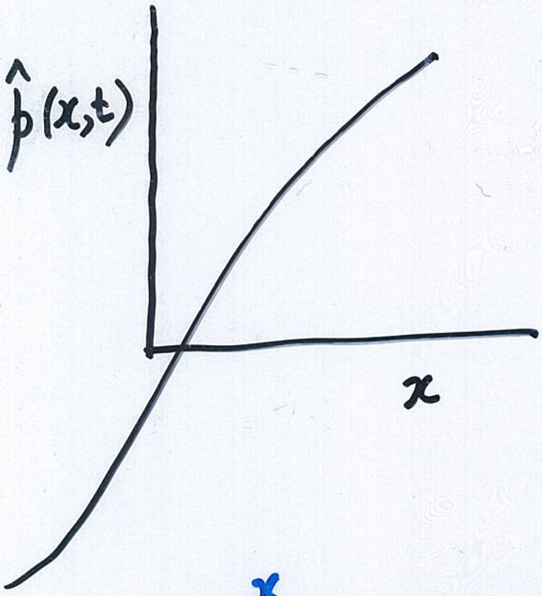
Momentum Measurement by going to $t\to\infty$

i.e., $\rho(x,p,t) \Rightarrow |\langle x|\psi(t)\rangle|^2\,\delta\!\left(p-\frac{mx}{t}\right)$

SOLUTION OF THIS PROBLEM FOR 2DIM PHASE SPACE: (SMR, VS : 1995)

INSTEAD OF ASSUMING $p = \frac{mx}{t}$, OR p_{dBB}

ASSUME ONLY THAT $p = \hat{p}(x,t)$, MONOTONIC FUNCTION OF x:



$$\rho(x,p,t) = |\psi(x,t)|^2 \, \delta\left(p - \hat{p}(x,t)\right)$$

REQUIRE

$$\int \rho(x,p,t)\, dx = |\langle p|\psi(t)\rangle|^2$$

THEN,

$$\int_{-\infty}^{x} dx' \, |\psi(x',t)|^2 = \int_{-\infty}^{\hat{p}} dp' \, |\tilde{\psi}(p',t)|^2$$

DETERMINES: $\hat{p}(x,t)$

LIOUVILLE PROPERTY OF $\rho(x,p,t)$, HAMILTONIAN EVOLUTION OF TRAJECTORIES CAN BE SHOWN.

Causal Hamiltonian $H_c(x,p,t)$

Just a Number [cf. Hamiltonian Differential Operator In Schröd. Eq^n]

MORE GENERAL ρ CAN BE CONSTRUCTED.

BUT OBSTRUCTION: PHASE SPACE BELL INEQS. IN HIGHER DIMENSIONS

ROY-SINGH CAUSAL Q.M

$$\rho(x,p,t) = |\psi(x,t)|^2 \, |\tilde{\psi}(p,t)|^2 \times$$

$$\delta\left(\int_{-\infty}^{x} dx' \, |\psi(x',t)|^2 - \int_{-\infty}^{p} dp' \, |\tilde{\psi}(p',t)|^2 \right)$$

vanishes at $p = \hat{p}(x,t)$

$$= |\psi(x,t)|^2 \, \delta\left(p - \hat{p}(x,t)\right)$$

EXAMPLE : GAUSSIAN FREE PARTICLE

$$\tilde{\psi}(p,t) = (\alpha\pi)^{-1/4} \exp\left[-\frac{(p-\beta)^2}{2\alpha} - i\frac{p^2}{2m} t \right]$$

$$|\psi(q,t)|^2 = (\alpha\pi)^{-1/2} \frac{m\alpha}{\sqrt{m^2+\alpha^2 t^2}} \times$$

$$\exp\left\{ -\frac{m^2\alpha}{m^2+\alpha^2 t^2} \left(q - \frac{\beta t}{m} \right)^2 \right\},$$

Then, $$\hat{p}(q,t) - \beta = \frac{\Delta p}{\Delta q}\left(q - \frac{\beta t}{m} \right),$$

where $(\Delta p)^2 = \frac{\alpha}{2}$, $(\Delta q)^2 = \dfrac{1 + \frac{\alpha^2 t^2}{m^2}}{2\alpha}$

CAUSAL QM FOR EPR STATE

Original EPR State

$$|\psi^{EPR}\rangle = |\hat{q}_1 - \hat{q}_2 = -q_0\rangle |\hat{p}_1 + \hat{p}_2 = p_0\rangle$$

Not normalizable.

Consider instead EPR-like state with free-particle time evolution:

$$\psi(\vec{q}, t) = \psi_1(q_1 - q_2, t)\, \psi_2\left(\frac{q_1+q_2}{2}, t\right),$$

$$\psi_1 = \frac{b_1^{1/2}}{\pi^{1/4}} \frac{1}{\sqrt{b_1^2 + \frac{2it}{m}}} \exp\left[-\frac{1}{2} \frac{(q_1 - q_2 + q_0)^2}{b_1^2 + \frac{2it}{m}}\right],$$

$$\psi_2 = (\pi b_2^2)^{-1/4} \sqrt{\frac{2mb_2^2}{2m + itb_2^2}} \exp f, \text{ where}$$

$$\times$$

$$f = -\frac{1}{2} \frac{b_2^2}{\left(1 + \left(\frac{b_2^2 t}{2m}\right)^2\right)}$$

$$\left[\left(\frac{q_1+q_2}{2} - \frac{p_0 t}{2m}\right)^2 - i\left\{\frac{b_2^2 t}{2m}\left(\frac{q_1+q_2}{2}\right)^2 + \frac{2p_0}{b_2^2}\frac{q_1+q_2}{2} - \frac{p_0^2 t}{2mb_2^2}\right\}\right],$$

which is EPR-like for b_1, b_2 small, $t=0$.

Since the c.m. motion separates, we can write a phase space density: (suppressing the time-dependence)

$$\rho(\vec{q}, \vec{p}) = \rho_1\left(q_1 - q_2, \frac{p_1 - p_2}{2}\right) \rho_2\left(\frac{q_1 + q_2}{2}, p_1 + p_2\right),$$

$$\rho_1 = |\psi_1(q_1 - q_2)|^2 \left|\tilde{\psi}_1\left(\frac{p_1 - p_2}{2}\right)\right|^2 \times$$

$$\delta\left[\int_{-\infty}^{q_1 - q_2} |\psi_1(q_1' - q_2')|^2 \, d(q_1' - q_2') - \int_{-\infty}^{p_1 - p_2} \left|\tilde{\psi}_1\left(\frac{p_1' - p_2'}{2}\right)\right|^2 d\left(\frac{p_1' - p_2'}{2}\right)\right],$$

$$\rho_2 = \left|\psi_2\left(\frac{q_1 + q_2}{2}\right)\right|^2 \left|\tilde{\psi}_2(p_1 + p_2)\right|^2 \times$$

$$\delta\left[\int_{-\infty}^{(q_1 + q_2)/2} \left|\psi_2\left(\frac{q_1' + q_2'}{2}\right)\right|^2 d\left(\frac{q_1' + q_2'}{2}\right) - \int_{-\infty}^{p_1 + p_2} |\tilde{\psi}_2(p_1' + p_2')|^2 \, d(p_1' + p_2')\right]$$

The position-momentum correlations can be calculated analytically for the EPR-like wave function.

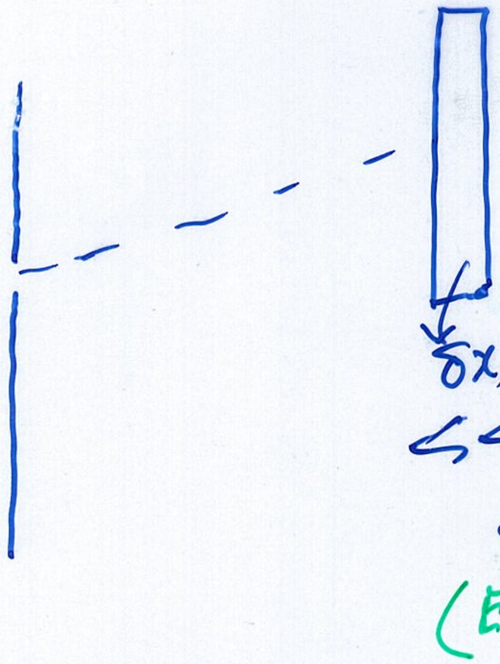
N.B. Q.M. probabs. for the 4 pairs: $\left(q_1 - q_2, \frac{q_1 + q_2}{2}\right)$, $(q_1 - q_2, p_1 + p_2)$, $\left(\frac{p_1 - p_2}{2}, p_1 + p_2\right)$, $\left(\frac{p_1 - p_2}{2}, \frac{q_1 + q_2}{2}\right)$ ARE REPRODUCED

IDEA FOR EXPTL TEST

ARBITRARILY ACCURATE SIMULT.
MEAS. OF q, p IMPOSSIBLE

BUT SIMULT. MEAS. WITH

$\hbar \ll \underbrace{\delta x \; \delta p}_{\text{inaccuracies}} \Leftarrow \underbrace{\Delta x \, \Delta p}_{\text{QUANTUM DISPERSIONS}}$



$\delta x \times \delta p \approx \hbar$

$\ll \Delta x \, \Delta p$

increases with time

(EXPANDING WAVE PACKETS)

Not possible so far: only quantum-optical analogues attempted

Compare with theory of Jt. measurements

Exptl. tests of Causal QM Joint Probabs. for EPR particle states possible because

$$\delta p << \Delta p$$

and

$$\delta x \sim \frac{\hbar}{\delta p} << \Delta x(t)$$

CAN BE CONSISTENT DUE TO EXPANDING WAVE PACKETS FOR $\Delta x(t)$

QUANTUM OPTICAL TESTS ?

We construct generalized coherent states with $\Delta q \, \Delta p = n + \frac{1}{2}$ and their EPR-LIKE ENTANGLED cousins for which

$$\delta p << \Delta p = \sqrt{n + \tfrac{1}{2}}, \text{ and}$$

$$\delta q \sim \frac{1}{\delta p} << \Delta q = \sqrt{n + \tfrac{1}{2}}$$

ARE CONSISTENT

COHERENT STATES WITH $\Delta q \Delta p = (n + \frac{1}{2}) \hbar$

$$H\psi = i\hbar \frac{\partial \psi}{\partial t}, \quad H = \hbar \omega a^{+} a$$

$$\frac{1}{2}(q^2 + p^2)\psi = i \frac{\partial \psi}{\partial \tau}, \quad a = \frac{q + ip}{\sqrt{2}}$$

$$\tau = \omega t, \quad a^{+} = \frac{q - ip}{\sqrt{2}}$$

USUAL COHERENT STATES

$$a|\alpha(t)\rangle = \alpha(t)\,|\alpha(t)\rangle, \quad \alpha = A e^{-i(\omega t + \phi)}$$

$$\psi_\alpha = \langle q | \alpha(t) \rangle = \phi_0(q - q_{cl}) \exp\left[i \frac{p_{cl}}{\hbar} \left(q - \frac{1}{2} q_{cl}\right) - i \frac{\tau}{2} \right]$$

SHAPE PRESERVING WAVE PACKET, $\Delta q \Delta p = \frac{1}{2}$

$\phi_0(q)$: GROUND STATE WAVE FUNCTION

$$q_{cl} \equiv \mathrm{Re}\,\alpha, \quad p_{cl} \equiv \mathrm{Im}\,\alpha = \frac{dq_{cl}}{d\tau}$$

GENERALIZED COHERENT STATES

(SMR & V.S. P.R.D25, 3413 (1982))

$$(a^{+} - \alpha^{*})(a - \alpha(t))\,|n,\alpha\rangle = n\,|n,\alpha\rangle$$

$$\psi_{n,\alpha} = \langle q|n,\alpha\rangle$$
$$= \phi_n(q - q_{cl})\exp\left[i p_{cl}\left(q - \tfrac{1}{2}q_{cl}\right) - i\left(n+\tfrac{1}{2}\right)\tau\right],$$

$$\phi_n(q) = e^{-q^2/2} H_n(q)/\sqrt{h_n},$$
$$h_n = \sqrt{\pi}\, 2^n\, n!$$

(EXCITED STATE WAVE FNS:
~ PARABOLIC CYLINDER FUNCTIONS OF $q\sqrt{2}$)

$|\psi_{n,\alpha}(\tau)|$ SHAPE PRESERVING WAVE PACKET, CLASSICAL MOTION,

BUT

$$\Delta q \Delta p = \left(n + \tfrac{1}{2}\right)$$

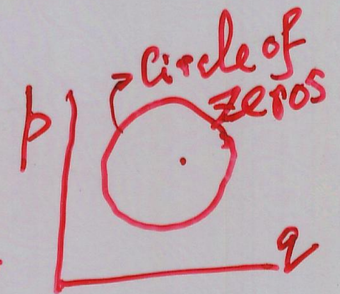
NON-CLASSICALITY FOR $n > 0$ SHOWN BY NON-POSITIVE DEFINITENESS OF WIGNER FUNCTION (Q. TOMOGRAPHY): PRESENT WORK

Q. TOMOGRAPHY:

WIGNER FN. OF GENERALIZED COHERENT STATE

$$W_{n,\alpha}(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipy}\, dy \, \langle q-\tfrac{1}{2}y|\psi_{n,\alpha}\rangle\langle\psi_{n,\alpha}|q+\tfrac{1}{2}y\rangle$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{i(p-p_{cl})y}\,\phi_n(q-q_{cl}-\tfrac{1}{2}y)\,\phi_n(q-q_{cl}+\tfrac{1}{2}y)$$



$$\xrightarrow{n=0} \left[\frac{1}{\pi} e^{-(q-q_{cl})^2-(p-p_{cl})^2}\right]$$

$$\xrightarrow{n=1} 2\left[\quad \text{''} \quad\right]\left[(q-q_{cl})^2+(p-p_{cl})^2-\tfrac{1}{2}\right]$$

$$\xrightarrow{n=2} 2\left[\quad \text{''} \quad\right]\left[\left\{(q-q_{cl})^2+(p-p_{cl})^2-1\right\}^2-\tfrac{1}{2}\right]$$

$$\xrightarrow{n=3} \frac{e^{-|z|^2}}{3\pi}\left[4|z|^6-18|z|^4+18|z|^2-3\right],$$

$$|z|^2 = (q-q_{cl})^2+(p-p_{cl})^2.$$

Zeros at $|z|^2 \simeq 0.2079,\ 1.1471,\ 3.145$

HAS n CONCENTRIC CIRCLES OF REAL ZEROS (?) IN ~~$|z|^2$~~ PHASE SPACE

EXPLICIT NON-CLASSICALITY

ENTANGLED GENERALIZED COHERENT STATES FOR 2-MODE LIGHT

$$H = \hbar\omega\left(a_1^{+} a_1 + a_2^{+} a_2\right)$$

$$= \frac{1}{2}\hbar\omega\left(q_1^2 + p_1^2 + q_2^2 + p_2^2\right)$$

$$= \frac{1}{4}\hbar\omega\left[(q_1+q_2)^2 + (p_1+p_2)^2 + (q_1-q_2)^2 + (p_1-p_2)^2\right].$$

$$\left[\frac{1}{\sqrt{2}}(q_1 \pm q_2), \frac{1}{\sqrt{2}}(p_1 \pm p_2)\right] = i$$

$$\left[\frac{1}{\sqrt{2}}(q_1 \pm q_2), \frac{1}{\sqrt{2}}(p_1 \mp p_2)\right] = 0$$

$\therefore$ $H\psi = i\hbar\dfrac{\partial\psi}{\partial t}$ has a solution:

(PRODUCT OF SHAPE-PRESERVING LUMPS)

$$\psi = \psi_{m,\alpha}\left(\frac{q_1+q_2}{\sqrt{2}}, \tau\right)\psi_{n,\beta}\left(\frac{q_1-q_2}{\sqrt{2}}, \tau\right)$$

$\tau = \omega t$, $\alpha = A e^{-i(\omega t+\phi)}$, $\beta = B e^{-i(\tau+\phi')}$,

m, n integers (non-negative)

$$\Delta(q_1+q_2)\Delta(p_1+p_2) = 2\left(m+\frac{1}{2}\right)$$

$$\Delta(q_1-q_2)\Delta(p_1-p_2) = 2\left(n+\frac{1}{2}\right)$$

WIGNER FUNCTIONS OF

$$\psi = \left\langle \frac{q_1+q_2}{\sqrt{2}}, \middle| m\alpha \right\rangle \left\langle \frac{q_1-q_2}{\sqrt{2}} \middle| n\beta \right\rangle$$ ARE:

$$W_{m,\alpha}\left(\frac{q_1+q_2}{\sqrt{2}}, \frac{p_1+p_2}{\sqrt{2}}\right) W_{n,\beta}\left(\frac{q_1-q_2}{\sqrt{2}}, \frac{p_1-p_2}{\sqrt{2}}\right)$$

ZEROS ON 3-DIM SURFACES IN PHASE SPACE CORRESPONDING TO ONE OF THE FACTORS VANISHING

FUTURE : PHASE SPACE BELL INEQS FOR THESE STATES :

THEORY OF JOINT q, p MEASUREMENTS: ARTHURS-KELLY

$$H_{int} = K[q p_1 + p p_2], \quad [p_1, p_2] = 0$$

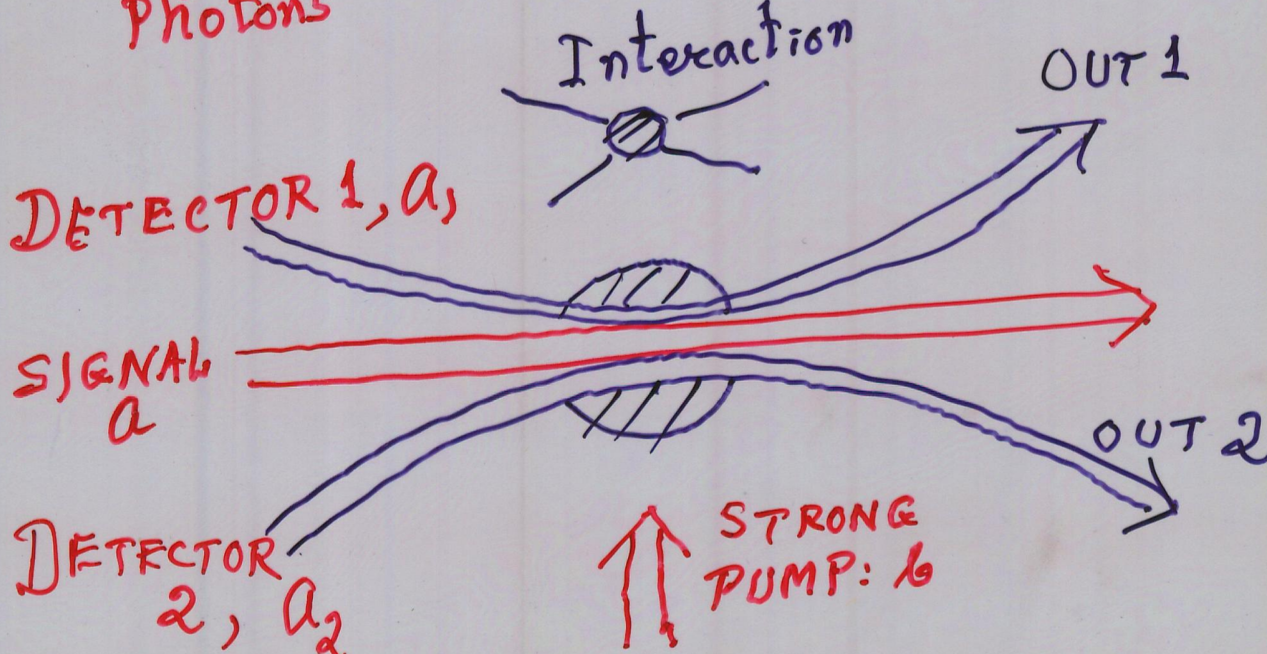
IDEA FOR QUANTUM-OPTICS EXPT

(e.g. S. Stenholm, Ann. Phys. 218, 233 (1992))

$$H_{int} = \frac{i}{2}\left[a^{+}(a_1^{+} + i a_2^{+}) b^2 + a (a_1^{+} - i a_2^{+}) b^{+} b - H.c.\right]$$

$b \to \beta$
Classical Limit of Strong Pump Photons

$$\beta^2 (q p_1 + p p_2)$$



VON-NEUMANN: IDEAL MEASUREMENT OF OBSERVABLE A:

$\{\phi_n(q)\}$ c.o.n. set (Eig. Fns. of A), $\phi = \sum c_n \phi_n$

System (I) + Meter (II): Hamiltonian H

Initial state $\phi(q)\,\chi(x) = \left(\sum_n c_n \phi_n(q)\right)\chi(x)$

$\xrightarrow[\text{evolution}]{\text{Hamilt.}} \sum_n c_n \phi_n(q)\,\chi_n(x)$

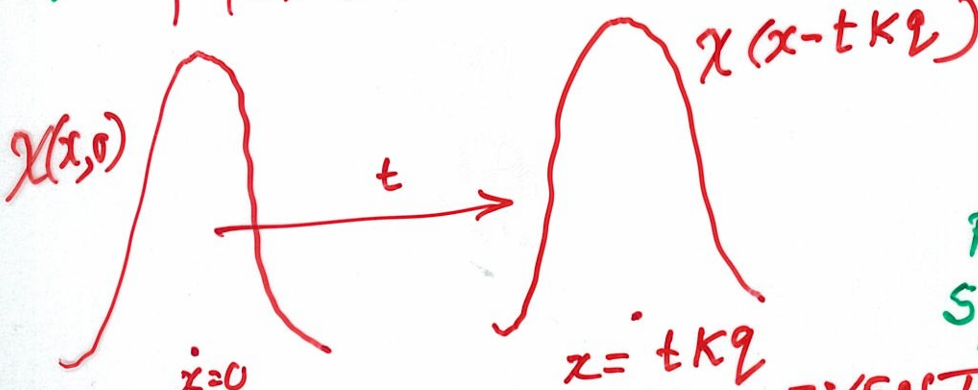
Tied together

INEXPLICABLE: "OBSERVATION" OF METER STATE → REVEALS SYSTEM STATE

CONCRETE MODEL TO MEASURE POSITION:

$i\dfrac{\partial \psi}{\partial t} = Kq\left(-i\dfrac{\partial}{\partial x}\right)\psi$, $\psi(t=0) = \phi(q)\chi(x)$

$\Rightarrow \psi(q,x,t) = \phi(q)\,\chi(x - tKq)$



If χ SHARPLY PEAKED, SHIFT OF PEAK REVEALS SYSTEM POSITION.

POSITION IN INDIVIDUAL EVENTS CAN BE MEASURED WITH ARBITRARY ACCURACY:

STATISTICAL DISPERSION: Δq

SIMULTANEOUS MEASUREMENT OF NON-COMMUTING OBSERVABLES.

E. ARTHURS & J. L. KELLY JR.
Bell System Technical Journal 44, 725 (1965)
E. Arthurs & M. S. Goodman, P.R.L. 60, 2447 (1988)
P. Busch, T. Heinonen, P. Lahti, Phys. Reports 452, 155-176 (2007)

VALUES OF q, p (non-commuting) tied by Schröd. Evolution to values of commuting meter observables x_1, x_2:

NEW UNCERTAINTY RELATION:

$$\Delta q \, \Delta p \geqslant \tfrac{1}{2}\hbar \Longrightarrow \Delta x_1 \Delta x_2 \geqslant \hbar$$

AFTER MEASUREMENT SYSTEM COLLAPSES TO STATE WITH MINM. UNCERTAINTY ($\Delta q \Delta p = \frac{1}{2}\hbar$) WITH $\langle q \rangle = x_1$, $\langle p \rangle = x_2$

MODEL: $H = K\left(q\left(-i\frac{\partial}{\partial x_1}\right) + p\left(-i\frac{\partial}{\partial x_2}\right)\right)$

$$\psi(t=0) = \phi(q)\,\chi_1(x_1)\,\chi_2(x_2)$$

$$\psi(q, x_1, x_2, t) \equiv \frac{1}{\sqrt{2\pi}} \int e^{ip_2 x_2}\, dp_2\, \tilde{\psi}(q, x_1, p_2, t)$$

In q, x_1, p_2 representation:

$$i\frac{\partial \tilde{\psi}(q, x_1, p_2, t)}{\partial t} = K\left(q(-i)\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial q} p_2\right)\tilde{\psi}$$

EXACT SOLUTION:

$$\tilde{\psi}(q, x_1, p_2, t) = \phi(q - Kt\, p_2) \times$$
$$\times\, \chi_1(x_1 - qKt + \tfrac{1}{2} p_2 K^2 t^2)\, \tilde{\chi}_2(p_2).$$

$$\psi(q, x_1, x_2, t) = \frac{1}{\sqrt{2\pi}} \int e^{i p_2 x_2}\, dp_2\, \tilde{\psi}(q, x_1, p_2, t)$$

$$P(x_1, x_2, t) = \int_{-\infty}^{\infty} dq\, |\psi(q, x_1, x_2, t)|^2$$

CHOOSE MINM. UNCERTAINTY METER STATES:

$$\chi_1(x_1) = \frac{1}{\pi^{1/4} b^{1/2}} \exp\left(-\frac{x_1^2}{2b^2}\right)$$

$$\tilde{\chi}_2(p_2) = \frac{1}{(4\pi)^{1/4} b^{1/2}} \exp\left(-\frac{p_2^2}{8b^2}\right)$$

THEN,

$$P(x_1, x_2, t = \tfrac{1}{K}) = \frac{1}{b\,(2\pi)^{3/2}} \left| \int_{-\infty}^{\infty} dq\, e^{-i x_2 q}\, \phi(q)\, e^{-\frac{1}{4b^2}(x_1 - q)^2} \right|^2$$

$$= \frac{b}{\sqrt{2}\, \pi^{3/2}} \left| \int_{-\infty}^{\infty} dp\, \tilde{\phi}(p)\, e^{-b^2 (p - x_2)^2}\, e^{i p x_1} \right|^2, \text{ AND,}$$

STATE AFTER MEASUREMENT VALUES x_1, x_2

$$\phi_{t=1/K}(q) = \psi(q, x_1, x_2, t = 1/K) \,/\, \|\psi\|$$

$$= \frac{1}{\sqrt{b}} \frac{1}{(2\pi)^{1/4}}\, e^{i q x_2 - \frac{1}{4b^2}(x_1 - q)^2}.$$

MINM. UNCERTAINTY STATE

$$\equiv \phi_{b, x_1, x_2}(q)$$

$$\tilde{\varphi}_{b,x_1,x_2}(p) = \frac{\sqrt{2b}}{(2\pi)^{3/4}} \exp\left[-b^2(p-x_2)^2 - ipx_1 + ix_1x_2\right]$$

(Fourier transform)

$$\Delta q = b, \quad \Delta p = \frac{1}{2b}$$

$$P(x_1, x_2, t = \tfrac{1}{\kappa}) = \frac{1}{2\pi} \underbrace{|\langle \varphi_{b,x_1,x_2} | \varphi \rangle|^2}_{\text{HUSIMI FUNCTION (1940)}}$$

$$= \int dq\,dp\, W_{\varphi_{b,x_1,x_2}}(q,p)\, W_\varphi(q,p),$$

SMEARED WIGNER FUNCTION

HUSIMI FUNCTION
(1940)

where

$$W_{\varphi_{b,x_1,x_2}}(q,p) = \frac{1}{\pi} \exp\left[-\frac{1}{2b^2}(q-x_1)^2 - 2b^2(p-x_2)^2\right]$$

$$\langle x_1 \rangle = \int dx_1\, x_1 \int dx_2\, P(x_1, x_2) = \langle q \rangle$$

$$\langle x_2 \rangle = \int dx_2\, x_2 \int dx_1\, P(x_1, x_2) = \langle p \rangle$$

BUT DISPERSIONS IN x_1, x_2 LARGER:

$$(\Delta x_1)^2 = (\Delta q)^2 + b^2 = \sigma_q^2 + b^2$$

$$(\Delta x_2)^2 = (\Delta p)^2 + \frac{1}{4b^2} = \sigma_p^2 + \frac{1}{4b^2}$$

EXTRA VARIANCE DUE TO DISTURBANCE CAUSED BY JOINT MEASUREMENT.

TO MINIMISE $\Delta x_1 \Delta x_2$, CHOOSE $b^2 = \frac{\sigma_q}{2\sigma_p}$

$$\Delta x_1 \Delta x_2 \geq \sigma_q \sigma_p + \frac{1}{2}$$

$$\geq 1$$

(WHEREAS $\sigma_q \sigma_p \geq 1/2$)

(MEASUREMENT UNCERTAINTY RELATION)

Does the maximum of $P(x_1, x_2)|_{\text{Arthurs-Kelly}}$

$$\frac{\partial P(x_1, x_2)}{\partial x_2} = 0, \text{ at } x_2 = \hat{x}_2(x_1)$$

coincide with the Roy-Singh formula?

$$\hat{x}_2 - \beta = \frac{\Delta p}{\Delta q}\left(x_1 - \frac{\beta t}{m}\right) \times$$

$$\left\{ \frac{(\Delta q)^2}{(\Delta q)^2 + b^2} \sqrt{1 - \frac{1}{4(\Delta q)^2 (\Delta p)^2}} \right\},$$

YES, in the limit $\frac{b^2}{(\Delta q)^2} \ll 1$

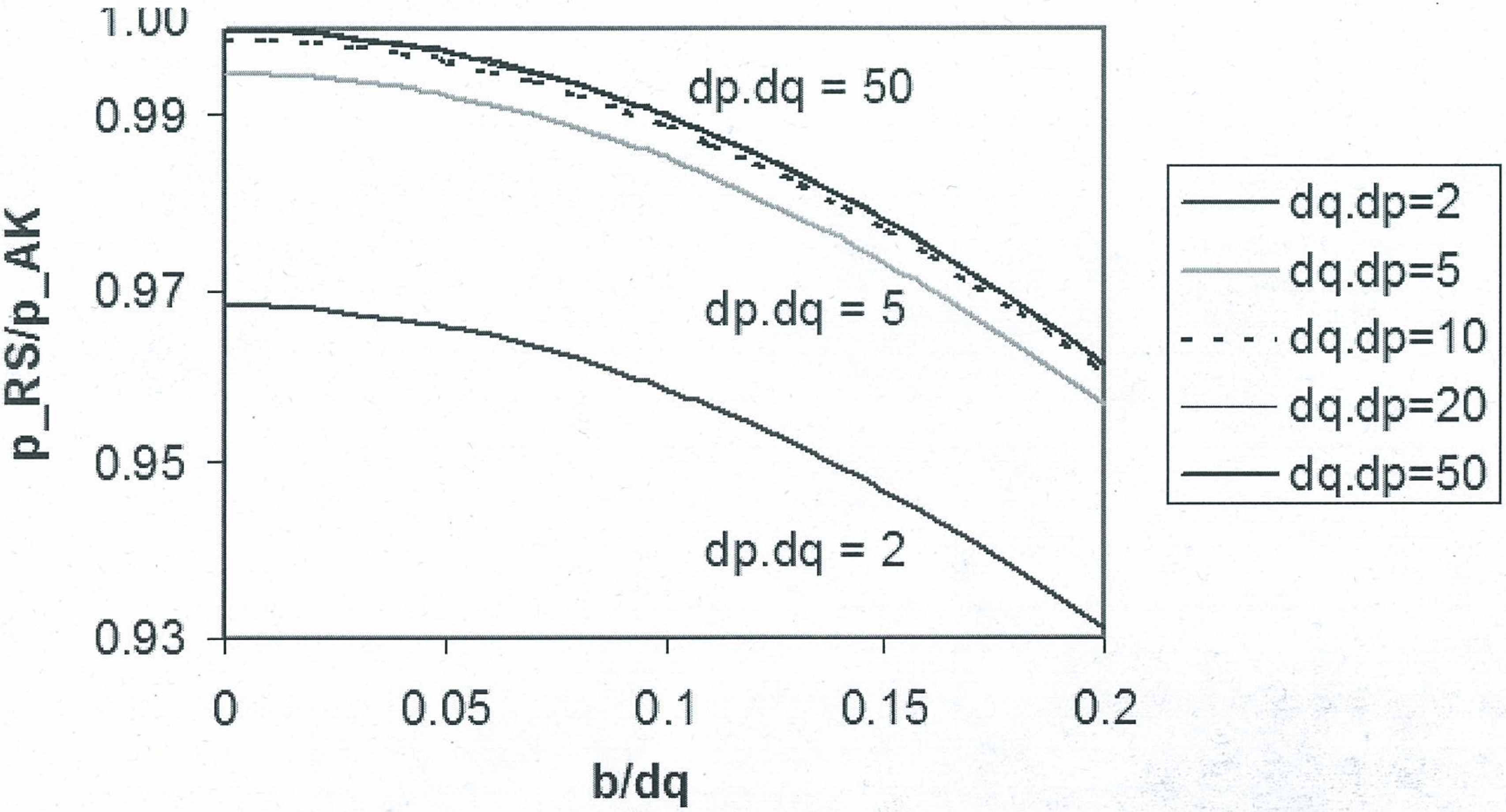
and $\frac{1}{4 b^2 (\Delta p)^2} \ll 1$, $\{\} \to 1$

giving exactly the Roy-Singh formula

(INSTRUMENTAL ERRORS
$\ll$ INTRINSIC SYSTEM DISPERSIONS)

POSSIBLE : $(\Delta p)^2 = \frac{\alpha}{2}$, $(\Delta q)^2 = \frac{1 + \frac{\alpha^2 t^2}{m^2}}{2\alpha}$

Need $\frac{1}{b^2 \alpha} \ll 1$, $\alpha b^2 \ll \left(\frac{\alpha t}{m}\right)^2$



The ratio R1 of the momentum in the Roy-Singh causal theory of quantum mechanics and the most probable value of the momentum in the Arthurs-Kelly theory for joint measurement of position and momentum is plotted versus the ratio R2 of the position measurement precision b to the position width of a Gaussian wave packet. There is excellent agreement between the two theories in the relevant region, the ratio R1 being between 1 and 0.99 for R2 < 0.1 and $dq\,dp > 10$. (FIG. COURTESY ARUNABHA S. ROY)

Causal Q.M. Reproducing $|\langle x|\psi\rangle|^2, |\langle p|\psi\rangle|^2$:

S. M. Roy & V. Singh,
Mod. Phys. Lett A10, 709 (1995) : 1 dim
Phys. Lett. A 255, 201 (1999) : N dim.
S. M. Roy, Pramana 59, 337 (2002): "Maximally classical"

Complementarity Principle For Quantum Probabilities : (Marginal Theorems)

A. Martin & SM Roy,
Phys Lett B 350, 66 (1995)

G. Auberson, G. Mahoux, SM Roy & V. Singh:
Phys Lett A 300, 327 (2002)
J. M. P. 44, 2729-2747 (2003)
J. M. P. 45, 4832-4854 (2004)

Separability Ineqs. For n particles:
S. M. Roy, in preparation & PRL 94, 010402 (2005)

Compatibility Conditions on Sub-system Density Matrices to be marginals of a single Density operator for the whole system:

A. Higuchi, A. Sudbery and J. Szulc, PRL 90, 107902 (2003)

P. Butterley, A. Sudbery and J. Szulc, quant-ph/0407227

S. M. Roy, in preparation

ARTHURS-KELLY JOINT MEASUREMENT

$x_1 \leftrightarrow q \,, \quad x_2 \leftrightarrow p$

$$\int dx_2\, P(x_1, x_2, t) = \frac{1}{b\sqrt{2\pi}} \int_{-\infty}^{\infty} dq\, |\varphi(q,t)|^2\, e^{-\frac{1}{2b^2}(x_1-q)^2}$$

$$\xrightarrow[\frac{b}{\Delta q} \to 0]{} |\varphi(x_1, t)|^2 ,$$

Since $\frac{1}{\lambda\sqrt{\pi}}\, e^{-(x_1-q)^2/\lambda^2} \xrightarrow[\lambda \to 0]{} \delta(x_1 - q)$

Similarly

$$\int dx_1\, P(x_1, x_2, t) \xrightarrow[b\Delta p \to \infty]{} |\tilde{\varphi}(x_2, t)|^2$$

IN FACT, FOR GENERAL φ [NOT JUST THE GAUSSIAN FREE PARTICLE], FOR $\frac{b}{\Delta q} \to 0$ AND $b\Delta p \to \infty$,

$$\int dx_2\, P_{\text{ARTHURS-KELLY}}(x_1, x_2, t) \to \int dx_2\, \rho_{\text{ROY-SINGH}}(x_1, x_2, t)$$

$$\int dx_1\, P_{\text{ARTHURS-KELLY}}(x_1, x_2, t) \to \int dx_1\, \rho_{\text{ROY-SINGH}}(x_1, x_2, t)$$